Completeness of Epistemic Coalition Logic with Group Knowledge

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Abstract

Coalition logic is one of the most popular logics for multi-agent systems. While epistemic extensions of coalition logic have received much attention, existence of their complete axiomatisations has so far been an open problem. In this paper we settle several of those problems. We prove completeness for epistemic coalition logic with common knowledge, with distributed knowledge, and with both common and distributed knowledge, respectively.

1 Introduction

Logics of coalitional ability such as *Coalition Logic* ($\mathcal{CL}$) [11], *Alternating-time Temporal Logic* ($\mathcal{ATL}$) [1], and STiT logics [2], are arguably some of the most studied logics in multi-agent systems in recent years. Many different variants of these logics have been proposed and studied, but so far meta-logical results have focused more on computational expressiveness and expressive power and less on completeness, with Goranko's and van Drimmelen's completeness proof for $\mathcal{ATL}$ [6], Pauly's completeness proof for $\mathcal{CL}$ [11] and Broersen and colleagues' completeness proofs for different variants of STiT logic [4, 3, 9] being notable exceptions.

The main construction in coalitional ability logics is of the form $[G]\phi$, where G is a set of agents and ϕ a formula, intuitively meaning that G is *effective* for ϕ, or that G can make ϕ come true no matter what the other agents do. One of the most studied extension of basic coalitional ability logics is adding *knowledge* operators of the type found in *epistemic logic* [5, 10]: both *individual* knowledge operators K_i where i is an agent, and different types of *group* knowledge operators E_G, C_G and D_G where G is a group of agents, standing for everybody-knows, common knowledge and distributed knowledge, respectively. Combining coalitional ability operators and epistemic operators in general and group knowledge operators in particular lets us express many potentially interesting properties of multi-agent systems, such as [12]:

- $K_i\phi \rightarrow [\{i\}]K_j\phi$: i can communicate her knowledge of ϕ to j

- $C_G\phi \to [G]\psi$: common knowledge in G of ϕ is sufficient for G to ensure that ψ
- $[G]\psi \to D_G\phi$: distributed knowledge in G of ϕ is necessary for G to ensure that ψ
- $D_G\phi \to [G]E_G\phi$: G can cooperate to make distributed knowledge explicit

In this paper we study a complete axiomatisation of variants of *epistemic coalition logic* ($\mathcal{ECL}$), extensions of coalition logic with individual knowledge and different combinations of common knowledge and distributed knowledge. Coalition logic, the next-time fragment of $\mathcal{ATL}$, is one of the most studied coalitional ability logics, and this paper settles a key open problem: completeness of its epistemic variants.

While epistemic coalitional ability logics have been studied to a great extent, we are not aware of any published completeness results for such logics with all epistemic operators. [12] gives some axioms of $\mathcal{ATEL}$, $\mathcal{ATL}$ extended with epistemic operators, but does not attempt to prove completeness[1]. Broersen and colleagues [3, 9] prove completeness of variants of STiT logic that include individual knowledge operators, but not group knowledge operators, and [9] concludes that adding group operators is an important challenge.

The rest of the paper is organised as follows. In the next section we first give a brief review of coalition logic, and how it is extended with epistemic operators. We then, in each of the three following sections, consider basic epistemic coalition logic with individual knowledge operators extended with common knowledge, with distributed knowledge, and with both common and distributed knowledge, respectively. For each of these cases we show a completeness result. For the common knowledge case we also show a filtration result. We conclude in Section 6.

2 Background

We define several extensions of propositional logic, and the usual derived connectives, such as $\phi \to \psi$ for $\neg\phi \wedge \psi$, will be used.

2.1 Coalition Logic

Assume a set Θ of atomic propositions, and a finite set N of agents. A *coalition* is a set $G \subseteq N$ of agents. We sometimes abuse notation and write a singleton coalition $\{i\}$ as i.

[1] In an abstract of a talk given at the LOFT workshop in 2004 [7], the authors propose a full axiomatisation of $\mathcal{ATEL}$ with individual knowledge and common knowledge operators with similar axioms. However, neither a completeness proof nor the result itself was published, and a proof indeed does not exist as explained to us in personal communication (Valentin Goranko).

The language of coalition logic ($\mathcal{CL}$) is defined by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [G]\phi$$

where $p \in \Theta$ and $G \subseteq N$.

A *coalition model* is a tuple

$$M = \langle S, E, V \rangle$$

where

- S is a non-empty set of *states*;
- V is a *valuation function*, assigning a set $V(s) \subseteq \Theta$ to each state $s \in S$;
- E assigns a *truly playable effectivity function* $E(s)$ to each state $s \in S$.

An *effectivity function* [11] over N and a set of states S is a function E that maps any coalition $G \subseteq N$ to a set of sets of states $E(G) \subseteq 2^S$. An effectivity function is *truly playable* [11, 8] iff it satisfies the following conditions:

E1 $\forall s \in S \forall G \subseteq N\ \emptyset \notin E(G)(s)$ (Liveness)

E2 $\forall s \in S \forall G \subseteq N\ S \in E(G)(s)$ (Safety)

E3 $\forall s \in S \forall X \subseteq S\ \bar{X} \notin E(\emptyset)(s) \Rightarrow X \in E(N)(s)$ (N-maximality)

E4 $\forall s \in S \forall G \subseteq N \forall X \subseteq Y \subseteq S\ X \in E(G)(s) \Rightarrow Y \in E(G)(s)$ (outcome monotonicity)

E5 $\forall s \in S \forall G_1, G_2 \subseteq N \forall X, Y \subseteq S\ X \in E(G_1)(s)$ and $Y \in E(G_2)(s) \Rightarrow X \cap Y \in E(G_1 \cup G_2)(s)$, where $G_1 \cap G_2 = \emptyset$ (superadditivity)

E6 $E^{nc}(\emptyset) \neq \emptyset$, where $E^{nc}(\emptyset)$ is the *non-monotonic core* of the empty coalition, namely

$$\{X \in E(\emptyset) : \neg\exists Y (Y \in E(\emptyset) \text{ and } Y \subset X)\}$$

An effectivity function that only satisfies E1-E5 is called *playable*. On finite domains an effectivity function is playable iff it is truly playable [8], because on finite domains E6 follows from E1-E5.

An $\mathcal{CL}$ formula is interpreted in a state in a coalition model as follows:

$M, s \models p$ iff $p \in V(s)$

$M, s \models \neg\phi$ iff $M, s \not\models \phi$

$M, s \models (\phi_1 \wedge \phi_2)$ iff ($M, s \models \phi_1$ and $M, s \models \phi_2$)

$M, s \models [G]\phi$ iff $\phi^M \in E(s)(G)$

where $\phi^M = \{t \in S : M, t \models \phi\}$.

Figure 1 shows an axiomatisation CL of coalition logic which is sound and complete wrt. all coalition models [11]. The following *monotonicity rule* is derivable, and will be useful later: $\vdash_{CL} \phi \rightarrow \psi \Rightarrow \vdash_{CL} [G]\phi \rightarrow [G]\psi$.

Prop Classical propositional logic

G1 $\neg[G]\bot$

G2 $[G]\top$

G3 $\neg[\emptyset]\neg\phi \rightarrow [N]\phi$

G4 $[G](\phi \wedge \psi) \rightarrow [G]\psi$

G5 $[G_1]\phi \wedge [G_2]\psi \rightarrow [G_1 \cup G_2](\phi \wedge \psi)$, if $G_1 \cap G_2 = \emptyset$

MP $\vdash_{CL} \phi, \phi \rightarrow \psi \Rightarrow \vdash_{CL} \psi$

RG $\vdash_{CL} \phi \leftrightarrow \psi \Rightarrow \vdash_{CL} [G]\phi \leftrightarrow [G]\psi$

Figure 1: CL: axiomatisation of $\mathcal{CL}$.

2.2 Adding Knowledge Operators

Epistemic extensions of coalition logic were first proposed in [12][2]. They are obtained by extending the language with *epistemic operators*, and the models with *epistemic accessibility relations*.

An epistemic accessibility relation for agent i over a set of states S is a binary relation $\sim_i \subseteq S \times S$. We will assume that epistemic accessibility relations are equivalence relations. An *epistemic coalition model*, henceforth often called simply a *model*, is a tuple

$$M = \langle S, E, \sim_1, \ldots, \sim_n, V\rangle$$

where $\langle S, E, V\rangle$ is a coalition model and $\sim_i$ is an epistemic accessibility relation over S for each agent i.

Epistemic operators come in two types: individual knowledge operators K_i, where i is an agent, and group knowledge operators C_G and D_G where G is a coalition for expressing *common knowledge* and *distributed knowledge*, respectively. Formally, the language of $\mathcal{CLCD}$ (*coalition logic with common and distributed knowledge*), is defined by extending coalition logic with all of these operators:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [H]\phi \mid K_i\phi \mid C_G\phi \mid D_G\phi$$

where $p \in \Theta$, $i \in N$, $H \subseteq N$ and $\emptyset \neq G \subseteq N$. When G is a coalition, we write $E_G\phi$ as a shorthand for $\bigwedge_{i \in G} K_i\phi$ (everyone in G knows ϕ).

The languages of the logics $\mathcal{CLK}$, $\mathcal{CLC}$ and $\mathcal{CLD}$ are the restrictions of this language with no C_G and no D_G operators, no D_G operators, and no C_G operators, respectively.

The interpretation of these languages in an (epistemic coalition) model M is defined by adding the following clauses to the definition for $\mathcal{CL}$:

[2] In that paper for $\mathcal{ATL}$; $\mathcal{CL}$ is a fragment of $\mathcal{ATL}$.

$M, s \models K_i\phi$ iff $\forall t \in S, (s,t) \in \sim_i \Rightarrow M, t \models \phi$

$M, s \models C_G\phi$ iff $\forall t \in S, (s,t) \in (\bigcup_{i \in G} \sim_i)^* \Rightarrow M, t \models \phi$

$M, s \models D_G\phi$ iff $\forall t \in S, (s,t) \in \bigcap_{i \in G} \sim_i \Rightarrow M, t \models \phi$

where R^* denotes the transitive closure of the relation R. We use $\models \phi$ to denote the fact that ϕ is *valid*, i.e., that $M, s \models \phi$ for all M and states s in M.

2.2.1 Some Auxiliary Definitions

The following are some auxiliary concepts that will be useful in the following.

A *pseudomodel* is a tuple $M = (S, \{\sim_i : i \in N\}, \{R_G : \emptyset \neq G \subseteq N\}, E, V)$ where $(S, \{\sim_i : i \in N\}, E, V)$ is a model and:

- $R_G \subseteq S \times S$ is an equivalence relation for each G
- For any $i \in N$, $R_i = \sim_i$
- For any G, H, $G \subseteq H$ implies that $R_H \subseteq R_G$

The interpretation of a $\mathcal{CLCD}$ formula in a state of a pseudomodel is defined as for a model, except for the case for D_G which is interpreted by the R_G relation:

$M, s \models D_G\phi$ iff $\forall t \in S, (s,t) \in R_G \Rightarrow M, t \models \phi$

An *epistemic model* is a model without the E function, i.e., a tuple $\langle S, \sim_1, \ldots, \sim_n, V\rangle$. An *epistemic pseudomodel* is a pseudomodel without the E function, i.e., a tuple $\langle S, \{\sim_i : i \in N\}, \{R_G : \emptyset \neq G \subseteq N\}, V\rangle$.

Finally, a *playable (pseudo)model* is a (pseudo)model where only conditions E1-E5 on E hold.

3 Coalition Logic with Common Knowledge

In this section we consider the logic $\mathcal{CLC}$, extending coalition logic with individual knowledge operators and common knowledge. We first prove a completeness result, and then show that $\mathcal{CLC}$ admits filtrations.

3.1 Completeness

The axiomatisation CLC is shown in Figure 2. It extends CL with standard axioms and rules for individual and common knowledge (see, e.g., [5]).

It is easy to show that CLC is sound wrt. all models.

Lemma 1 (Soundness) *For any CLC-formula ϕ, $\vdash_{CLC} \phi \Rightarrow \models \phi$.*

In the remainder of this section we show that CLC also is complete.

Theorem 1 *Any CLC-consistent formula is satisfied in some model.*

Prop Classical propositional logic

G1 $\neg[G]\bot$

G2 $[G]\top$

G3 $\neg[\emptyset]\neg\phi \to [N]\phi$

G4 $[G](\phi \wedge \psi) \to [G]\psi$

G5 $[G_1]\phi \wedge [G_2]\psi \to [G_1 \cup G_2](\phi \wedge \psi)$, if $G_1 \cap G_2 = \emptyset$

MP $\vdash_{CLC} \phi, \phi \to \psi \Rightarrow \vdash_{CLC} \psi$

RG $\vdash_{CLC} \phi \leftrightarrow \psi \Rightarrow \vdash_{CLC} [G]\phi \leftrightarrow [G]\psi$

K $K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$

T $K_i\phi \to \phi$

4 $K_i\phi \to K_iK_i\phi$

5 $\neg K_i\phi \to K_i\neg K_i\phi$

C1 $E_G\phi \leftrightarrow \bigwedge_{i \in G} K_i\phi$

C2 $C_G\phi \to E_G(\phi \wedge C_G\phi)$

RN $\vdash_{CLC} \phi \Rightarrow \vdash_{CLC} K_i\phi$

RC $\vdash_{CLC} \phi \to E_G(\phi \wedge \psi) \Rightarrow \vdash_{CLC} \phi \to C_G\psi$

Figure 2: CLC: axiomatisation of $\mathcal{CLC}$.

Proof We define a canonical playable model $M^c = (S^c, \{\sim_i^c : i \in N\}, E^c, V^c)$ as follows:

S^c is the set of all maximally CLC consistent sets of formulas

$s \sim_i^c t$ iff $\{\psi : K_i\psi \in s\} = \{\psi : K_i\psi \in t\}$

$$X \in E^c(G)(s) \text{ iff } \begin{cases} \exists\phi\{s \in S^c : \phi \in s\} \subseteq X : [G]\phi \in s & G \neq N \\ \forall\phi\{s \in S^c : \phi \in s\} \subseteq S^c \setminus X : [\emptyset]\phi \notin s & G = N \end{cases}$$

V^c: $s \in V^c(p)$ iff $p \in s$

The conditions on $\sim_i^c$ (that it is an equivalence relation) and on E^c (that it satisfies E1-E5) hold in M^c. The proof for $\sim_i^c$ is obvious and the proof for E^c is standard. The intuition of cause is that a formula belongs to a state s in a model iff it is true there (truth lemma). However, the canonical model is in general *not* guaranteed to satisfy every consistent formula in the $\mathcal{CLC}$ language; the case of C_G in the truth lemma does not necessarily hold. Therefore we

are going to transform M^c by filtration into a finite model for a given CLC consistent formula ϕ. Note that since ϕ is consistent, it will belong to at least one s in M^c.

Let $cl(\phi)$ be the set of subformulas of ϕ closed under single negations and the condition that $C_G\psi \in cl(\phi) \Rightarrow K_iC_G\psi \in cl(\phi)$ for all $i \in G$. We are going to filtrate M^c through $cl(\phi)$. The resulting model $M^f = (S^f, \{\sim_i^f : i \in N\}, E^f, V^f)$ is constructed as follows:

S^f is $\{[s]_{cl(\phi)} : s \in S^c\}$ where $[s]_{cl(\phi)} = s \cap cl(\phi)$. We will omit the subscript $cl(\phi)$ in what follows for readability.

$[s] \sim_i^f [t]$ iff $\{\psi : K_i\psi \in [s]\} = \{\psi : K_i\psi \in [t]\}$

$V^f(p) = [V^c(p)]_{cl(\phi)}$ (where $[X]_{cl(\phi)} = \{[s] : s \in X\}$). Again we will omit the subscript for readability.

$X \in E^f(G)([s])$ iff $\{s' : \phi_X \in s'\} \in E^c(G)(s)$ where $\phi_X = \vee_{[t]\in X}\phi_{[t]}$ and $\phi_{[t]}$ is a conjunction of all formulas in $[t]$.

We now prove by induction on the size of θ that for every $\theta \in cl(\phi)$, $M^f, [s] \models \theta$ iff $\psi \in [s]$.

case $\theta = p$ trivial

case booleans trivial

case $\theta = K_i\psi$ assume $M^f, [s] \not\models K_i\psi$. The latter means there is a $[s']$ such that $[s] \sim_i^f [s']$ and $M^f, [s'] \not\models \phi$. By the inductive hypothesis $\phi \notin [s']$. Since $[s']$ is deductively closed wrt $cl(\phi)$ and $K_i\psi \in cl(\phi)$, also $K_i\psi \notin [s']$. $[s] \sim_i^f [s']$ means that $[s]$ and $[s']$ contain the same K_i formulas from $cl(\phi)$, hence $K_i\psi \notin [s]$.

Assume $M^f, [s] \models K_i\psi$. Then for all $[s']$ such that $[s] \sim_i^f [s']$, $M^f, [s'] \models \psi$. This means by the IH that $\psi \in [s']$ for all $[s'] \sim_i^f [s]$. Assume by contradiction that $K_i\psi \notin [s]$. Then $\phi_{[s]}$, where $\phi_{[s]}$ is the conjunction of all formulas in $[s]$, is consistent with $\neg K_i\psi$. If we write $\langle K_i \rangle$ for the dual of the K_i modality, this is equivalent to: $\phi_{[s]} \wedge \langle K_i \rangle \neg\psi$ is consistent. By forcing choices,

$$\phi_{[s]} \wedge \langle K_i \rangle \bigvee_{\neg\psi\in[t]} \phi_{[t]}$$

is consistent. By the distributivity of $\langle K_i \rangle$ over $\vee$,

$$\bigvee_{\neg\psi\in[t]} (\phi_{[s]} \wedge \langle K_i \rangle \phi_{[t]})$$

is consistent. So for some $[t]$ with $\neg\psi \in [t]$, $\phi_{[s]} \wedge \langle K_i \rangle \phi_{[t]}$ is consistent. We claim that $[s] \sim_i^f [t]$. If this is the case, we have a contradiction, since we assumed that $\psi \in [s']$ for all $[s'] \sim_i^f [s]$.

Proof of the claim: if $\phi_{[s]} \wedge \langle K_i \rangle \phi_{[t]}$ is consistent, then $[s] \sim_i^f [t]$. Suppose not $[s] \sim_i^f [t]$, that is there is a formula χ such that $K_i\chi \in [s]$ and $\neg K_i\chi \in [t]$ or vice versa. Then we have $K_i\chi \wedge \phi_{[s]} \wedge \langle K_i \rangle(\neg K_i\chi \wedge \phi_{[t]})$ is consistent, but since K_i is an S5 modality, this is impossible. Same for the case when $\neg K_i\chi \in [s]$ and $K_i\chi \in [t]$.

case $\theta = [G]\psi$

$M^f, [s] \models [G]\psi$ iff $\psi^{M^f} \in E^f(G)([s])$ iff $\{s' : (\vee_{[t]\in\psi^{M^f}} \phi_{[t]}) \in s'\} \in E^c(G)(s)$ iff (by the IH) $\{s' : (\vee_{\psi\in[t]}\phi_{[t]}) \in s'\} \in E^c(G)(s)$ iff(*) $\{s' : \psi \in s'\} \in E^c(G)(s)$ iff(**) $[G]\psi \in s$ iff (since $[G]\psi \in cl(\phi)$) $[G]\psi \in [s]$.

Proof of (*): assume S^f contains $n+k$ states, $[t_1], \ldots, [t_n]$ contain ψ and $[s_1], \ldots, [s_k]$ contain $\neg\psi$. Clearly, $\phi_{[t_1]} \vee \ldots \vee \phi_{[t_n]} \vee \phi_{[s_1]} \ldots \vee \phi_{[s_k]}$ is provably equivalent to $\top$. Consider $\vee_{\psi\in[t]}\phi_{[t]}$. It is provably equivalent to $(\psi \wedge \phi_{[t_1]}) \vee \ldots \vee (\psi \wedge \phi_{[t_n]})$. Since for every $[s_i]$ such that $\neg\psi \in [s_i]$, $(\psi \wedge \phi_{[s_i]})$ is provably equivalent to $\bot$,

$$(\psi \wedge \phi_{[t_1]}) \vee \ldots \vee (\psi \wedge \phi_{[t_n]})$$

is provably equivalent to

$$(\psi \wedge \phi_{[t_1]}) \vee \ldots \vee (\psi \wedge \phi_{[t_n]}) \vee (\psi \wedge \phi_{[s_1]}) \vee \ldots \vee (\psi \wedge \phi_{[s_k]})$$

which in turn is provably equivalent to

$$\psi \wedge (\phi_{[t_1]} \vee \ldots \vee \phi_{[s_k]})$$

which in turn is equivalent to $\psi \wedge \top$ hence to ψ. So in M^c, $\{s' : (\vee_{\psi\in[t]}\phi_{[t]}) \in s'\} = \{s' : \psi \in s'\}$.

Proof of (**): since we defined $X \in E^c(N)(s)$ to hold iff $S^c \setminus X \notin E^c(\emptyset)(s)$, it suffices to show the case that $G \neq N$. The direction to the left is immediate: if $[G]\psi \in s$ then $\{s' \in S^c : \psi \in s'\} \in E^c(G)(s)$ by definition. For the other direction assume that $\{s' \in S^c : \psi \in s'\} \in E^c(G)(s)$, i.e., there is some γ such that $\{s' \in S^c : \gamma \in s'\} \subseteq \{s' \in S^c : \psi \in s'\}$ and $[G]\gamma \in s$. It is easy to see that $\{s' \in S^c : \gamma \in s'\} \subseteq \{s' \in S^c : \psi \in s'\}$ implies that $\vdash \gamma \to \psi$, and by the monotonicity rule it follows that $[G]\psi \in s$.

case $\theta = C_G\psi$ The proof is similar to [13]. First we show that in M^f, if $C_G\psi \in cl(\phi)$, then $C_G\psi \in [s]$ iff every state on every $\sup_{i\in G} \sim_i^f$ path from $[s]$ contains ψ.

Suppose $C_G\psi \in [s]$. The proof is by induction on the length of the path. If the path is of 0 length, then clearly by deductive closure and by $\psi \in cl(\phi)$ we have $\psi \in [s]$. We also have $C_G\psi \in [s]$ by the assumption. IH: if $C_G\psi \in [s]$, then every state on every $\cup_{i\in G} \sim_i^f$ path of length n from $[s]$ contains ψ *and* $C_G\psi$. Inductive step: let us prove this for paths of length $n+1$. Suppose we have a path $[s] \sim_{i_1}^f [s_1] \ldots \sim_{i_n}^f [s_n] \sim_{i_{n+1}}^f [s_{n+1}]$. By the

IH, $\psi, C_G\psi \in [s_n]$. Since s_n is deductively closed and $K_{i_{n+1}}C_G\psi \in cl(\phi)$, we have $K_{i_{n+1}}C_G\psi \in [s_n]$. Since $[s_n] \sim^f_{i_{n+1}} [s_{n+1}]$ and the definition of $\sim^f_{i_{n+1}}$, $C_G\psi \in [s_{n+1}]$ and hence by reflexivity $\psi \in [s_{n+1}]$.

For the other direction, suppose that every state on every $\cup_{i\in G} \sim^f$ path from $[s]$ contains ψ. Prove that $C_G\psi \in [s]$. Let $S_{G,\psi}$ be the set of all $[t]$ such that every state on every $\cup_{i\in G} \sim^f$ path from $[t]$ contains ψ. Note that each $[t]$ is/corresponds to a finite set of formulas so we can write its conjunction $\phi_{[t]}$. Consider a formula

$$\chi = \bigvee_{[t]\in S_{G,\psi}} \phi_{[t]}$$

Similarly to [13] it can be proved that $\vdash_{CLC} \phi_{[s]} \to \chi$, $\vdash_{CLC} \chi \to \psi$ and $\vdash_{CLC} \chi \to E_G\chi$. And from that follows that $\vdash_{CLC} \phi_{[s]} \to C_G\psi$ hence $C_G\psi \in [s]$.

Now we prove that $M^f, [s] \models C_G\psi$ iff $C_G\psi \in [s]$. $C_G\psi \in [s]$ iff every state on every $\cup_{i\in G} \sim^f_i$ path from $[s]$ contains ψ iff for every $[t]$ reachable from $[s]$ by a $\cup_{i\in G} \sim^f_i$ path, $M^f, [t] \models \psi$ iff $M^f, [s] \models C_G\psi$.

❏

It is obvious that in M^f, $\sim_i$ are equivalence relations. So what remains to be proved is that E^f satisfies E1-E6. Since S^f is finite, it suffices to show E1-E5, which for finite sets of states entail E6.

Proposition 1 *M^f satisfies E1-E5.*

Proof E1 Note that $\phi_\emptyset$ is an empty disjunction, namely $\bot$.

$\emptyset \in E^f(G)([s])$ iff (by definition of E^f) $\{s' : \bot \in s'\} \in E^c(G)(s)$ iff $\emptyset \in E^c(G)(s)$. Since E^c satisfies $E1$, $\emptyset \notin E^f(G)([s])$.

E2 $S^f \in E^f(G)([s])$ iff $\{s' : \bigvee_{[t]\in S^f} \in s'\} \in E^c(G)(s)$ iff $S^c \in E^c(G)(s)$. Since E^c satisfies $E2$, $S^f \in E^f(G)([s])$.

E3 Let $\bar{X} \notin E^f(\emptyset)([s])$. Then $\{s' : \phi_{\bar{X}} \in s'\} \notin E^c(\emptyset)(s)$. Note that $\{s' : \phi_{\bar{X}} \in s'\}$ is the complement of $\{s' : \phi_X \in s'\}$, since $\phi_{\bar{X}} = \neg\phi_X$. Since E^c satisfies E3, this means that $\{s' : \phi_X \in s'\} \in E^c(N)(s)$. Hence $X \in E^f(N)([s])$.

E4 Let $X \subseteq Y \subseteq S^f$ and $X \in E^f(G)([s])$. Clearly $\vdash_{CLC} \phi_X \to \phi_Y$. Hence $\{s' : \phi_X \in s'\} \subseteq \{s' : \phi_Y \in s'\}$. Since $X \in E^f(G)([s])$, we have $\{s' : \phi_X \in s'\} \in E^c(G)(s)$. Since E^c satisfies E4, $\{s' : \phi_Y \in s'\} \in E^c(G)(s)$ so $Y \in E^f(G)([s])$.

E5 Let $X \in E^f(G_1)([s])$ and $Y \in E^f(G_2)([s])$ and $G_1 \cap G_2 = \emptyset$. So $\{s' : \phi_X \in s'\} \in E^c(G_1)(s)$ and $\{s' : \phi_Y \in s'\} \in E^c(G_2)(s)$ and since E^c satisfies E5, $\{s' : \phi_X \in s'\} \cap \{s' : \phi_Y \in s'\} \in E^c(G_2)(s)$. Note that

$\{s' : \phi_X \in s'\} \cap \{s' : \phi_Y \in s'\} = \{s' : (\vee_{[t]\in X}\phi_{[t]}) \in s'$ and $(\vee_{[t]\in Y}\phi_{[t]}) \in s'\}$

which is in turn the same as

$$\{s' : (\vee_{[t]\in X\cap Y}\phi_{[t]} \in s'\}$$

since $\{s' : (\vee_{[t]\in X\cap Y}\phi_{[t]} \in s'\} \in E^c(G_2)(s)$, $X \cap Y \in E^f(G_1)([s])$.

❏

Corollary 1 *For any CLC-formula ϕ, $\vdash_{CLC} \phi$ iff $\models \phi$.*

4 Completeness of Coalition Logic with Distributed Knowledge

In this section we consider the logic $\mathcal{CLD}$, extending coalition logic with individual knowledge operators and distributed knowledge.

The axiomatisation CLD is shown in Figure 3. It extends CL with standard axioms and rules for individual and distributed knowledge (see, e.g., [5]).

As usual, soundness can easily be shown.

Lemma 2 (Soundness) *For any CLD-formula ϕ, $\vdash_{CLD} \phi \Rightarrow \models \phi$.*

In the remainder of this section we show that CLD also is complete.

For a set of formulae s, let $K_a s = \{K_a\phi : K_a\phi \in s\}$ and $D_G s = \{D_G\phi : D_G\phi \in s\}$.

Definition 1 (Canonical Playable Pseudomodel) *The canonical playable pseudomodel $M^c = (S^c, \{\sim_i^c : i \in N\}, \{R_G^c : \emptyset \neq G \subseteq N\}, E^C, V^c)$ for $\mathcal{CLD}$ is defined as follows:*

- *S^c is the set of maximal consistent sets.*
- *$s \sim_i^c t$ iff $K_a s = K_a t$*
- *$sR_G t$ iff $D_H s = D_H t$ whenever $H \subseteq G$*
- *$V^c(p) = \{s \in S^c : p \in s\}$*
- *$X \in E^c(G)(s)$ iff* $\begin{cases} \exists\phi\{s \in S^c : \phi \in s\} \subseteq X : [G]\phi \in s & G \neq N \\ \forall\phi\{s \in S^c : \phi \in s\} \subseteq S^c \setminus X : [\emptyset]\phi \notin s & G = N \end{cases}$

Lemma 3 (Pseudo Truth Lemma) *$M^c, s \models \phi \Leftrightarrow \phi \in s$.*

Proof The proof is by induction on ϕ. The epistemic cases are exactly as for standard normal modal logic. The case for coalition operators is exactly as in [11]. ❏

It is easy to check that $\sim_i^c$ are equivalence relations and E1-E5 hold for E^c.

Lemma 4 (Finite Pseudomodel) *Every CLD-consistent formula ϕ has a finite pseudomodel where E1-E6 hold.*

Proof The proof is exactly as in Theorem 1, namely the construction of M^f, but starting with a Canonical Playable Pseudomodel rather than Canonical Playable Model; the definition of M^c contains the clause

$$\Gamma R_G \Delta \text{ iff } \forall H \subseteq G\{\psi : D_H\psi \in \Gamma\} = \{\psi : D_H\psi \in \Delta\}$$

We add the following condition to the closure: $D_i\phi \in cl(\phi)$ iff $K_i\psi \in cl(\phi)$. We define M^f to be a pseudomodel instead of a model, by adding the clause:

$$[s]R_G^f[s'] iff \forall H \subseteq G\{\psi : D_H\psi \in [s]\} = \{\psi : D_H\psi \in [s']\}$$

We show that M^f is indeed a pseudomodel:

- $R_i^f = \sim_i^f$: this follows from the fact that $K_i\phi \in [s]$ iff $D_i\phi \in [s]$ for any i, ϕ and s, which holds because of the $K_i\phi \to D_i\phi$ axiom and the new closure condition above.
- $G \subseteq H \Rightarrow R_H^f \subseteq R_G^f$: this holds by definition.

We add a case for $\theta = D_G\psi$ to the inductive proof. This case is proven in exactly the same way as the $\theta = K_i\psi$ case: the definitions of $\sim_i^f$ and R_G^f are of exactly the same form (in particular, R_G^f is also an $S5$ modality). The proof that E1-E6 hold in the resulting pseudomodel is the same as in the proof of Theorem 1 for E^f. ❑

We are now going to transform the pseudomodel into a proper model; it is a well-known technique for dealing with distributed knowledge. In fact, we can make direct use of a corresponding existing result for epistemic logic with distributed knowledge, and extend it with the coalition operators/effectivity functions. We here give the more general result for the language with also common knowledge, which will be useful later.

Theorem 2 ([5]) *If $M = (S, \{\sim_i : i \in N\}, \{R_G : \emptyset \neq G \subseteq N\}, V)$ is an epistemic pseudomodel, then there is an epistemic model $M' = (S', \{\sim_i' : i \in N\}, V')$ and a surjective (onto) function $\mathbf{f} : S' \to S$ such that for every $s' \in S'$ and formula $\phi \in \mathcal{ELCD}$, $M, \mathbf{f}(s') \models \phi$ iff $M', s' \models \phi$.*

Proof This result is directly obtained from the completeness proof for $\mathcal{ELCD}$ sketched in [5, p. 70]. For a more detailed proof (for a more general language), see [14, Theorem 9]. ❑

Theorem 3 *If a formula is satisfied in a finite pseudomodel, then it is satisfied in a model.*

Proof Let $M = (S, \{\sim_i : i \in N\}, \{R_G : \emptyset \neq G \subseteq N\}, E, V)$ be a finite pseudomodel such that $M, s \models \phi$. Let $M_p = (S, \{\sim_i : i \in N\}, \{R_G : \emptyset \neq G \subseteq N\}, V)$ be the epistemic pseudomodel underlying M, and let $M'_p = (S', \{\sim'_i : i \in N\}, V')$ and $\mathbf{f} : S' \to S$ be as in Theorem 2. Let $\mathbf{image}(X) = \{s' : \mathbf{f}(s') \in X\}$ for any set $X \subseteq S$. Finally, let $M' = (S', \{\sim'_i : i \in N\}, E', V')$ where E' is defined as follows:

- For $G \neq N$:

$$Y \in E'(G)(u) \;\Leftrightarrow\; \exists X \subseteq S,\ (Y \supseteq \mathbf{image}(X) \text{and } X \in E(G)(\mathbf{f}(u)))$$

- for $G = N$:

$$Y \in E'(G)(u) \;\Leftrightarrow\; \bar{X} \notin E'(\emptyset)(u)$$

Two things must be shown: that M' is a proper model, and that it satisfies ϕ.

Since M'_p is an epistemic model, to show that M' is a model all that remains to be shown is that E' is truly playable. We now show that that follows from true playability of E.

E1 Note that $\mathbf{image}(X) = \emptyset$ iff $X = \emptyset$.

For $G \neq N$, $\emptyset \in E'(G)(u)$ iff (by definition of E') $\exists X \subseteq S, s \in S$ ($\emptyset \supseteq \mathbf{image}(X)$and $X \in E(G)(\mathbf{f}(u))$) iff $\emptyset \in E(G)(\mathbf{f}(u)))$ which is impossible since M satisfies E1. Note that in particular this proves $\emptyset \notin E'(\emptyset)(u)$.

For N, $\emptyset \in E'(N)(u)$ iff $S' \notin E'(\emptyset)(u)$ and we'll see that this is impossible below.

E2 Note that $\mathbf{image}(S) = S'$.

For $G \neq N$, $S' \in E'(G)(u)$ iff (by definition of E') $\exists X \subseteq S$, ($S' \supseteq \mathbf{image}(X)$and $X \in E(G)(\mathbf{f}(u))$) and since $S' \supseteq \mathbf{image}(S)$ and $S \in E(G)(\mathbf{f}(u))$, $S' \in E'(G)(u)$ holds. Note that in particular this proves $S' \in E'(\emptyset)(u)$.

For N, $S' \in E'(N)(u)$ iff $\emptyset \notin E'(\emptyset)(u)$ and this was proved above.

E3 $\forall u \in S' \forall Y \subseteq S'\ \bar{Y} \notin E'(\emptyset)(u) \Rightarrow Y \in E'(N)(u)$ follows immediately from the definition for $E'(N)$.

E4 E' is monotonic by definition for $G \neq N$.

For N, assume $X \subseteq Y$ and $X \in E'(N)(u)$. Then $\bar{X} \notin E'(\emptyset)(u)$. Since for $\emptyset$ we already have monotonicity and $\bar{Y} \subseteq \bar{X}$, $\bar{Y} \notin E'(\emptyset)(u)$. So $Y \in E'(N)(u)$.

E5 $\forall u \in S' \forall G_1, G_2 \subseteq N \forall X', Y' \subseteq S'\ X' \in E'(G_1)(u)$ and $Y' \in E'(G_2)(u)$ $\Rightarrow X' \cap Y' \in E'(G_1 \cup G_2)(u)$, where $G_1 \cap G_2 = \emptyset$

For $G_1, G_2 \neq N$:

Let $s = \mathbf{f}(u)$, $G_1 \cap G_2 = \emptyset$, $X' \in E'(G_1)(u)$, $Y' \in E'(G_2)(u)$. This means that for some X, Y, such that $X' \supseteq X$ and $Y' \supseteq Y$, $X \in E(G_1)(s)$ and $Y \in E(G_2)(s)$, so since M satisfies E5, $X \cap Y \in E(G_1 \cup G_2)(s)$. Since $X' \supseteq X$ and $Y' \supseteq Y$, $X' \cap Y' \supseteq X \cap Y$, so $X' \cap Y' \in E'(G_1 \cup G_2)(u)$.

For $G_1 = N$ (G_2 has to be $\emptyset$): $X' \in E'(N)(u)$, $Y' \in E'(\emptyset)(u)$, prove $X' \cap Y' \in E'(N)(u)$. We have $\bar{X}' \notin E'(\emptyset)(u)$, $Y' \in E'(\emptyset)(u)$. Assume by contradiction $X' \cap Y' \notin E'(N)(u)$, then $\overline{X' \cap Y'} \in E'(\emptyset)(u)$ which means $\bar{X}' \cup \bar{Y}' \in E'(\emptyset)(u)$. This together with $Y' \in E'(\emptyset)(u)$ and E5 for $G_1, G_2 = \emptyset$ gives $(\bar{X}' \cup \bar{Y}') \cap Y' \in E'(\emptyset)(u)$, that is $\bar{X}' \in E'(\emptyset)(u)$. The latter contradicts $\bar{X}' \notin E'(\emptyset)(u)$.

E6 Suppose $X \in E^{nc}(\emptyset)(s)$. We claim that for every u such that $s = \mathbf{f}(u)$, $\mathbf{image}(X) \in E^{'nc}(\emptyset)(u)$.

Assume by contradiction that there exists a $Y \subset \mathbf{image}(X)$ such that $Y \in E^{'nc}(\emptyset)(u)$ for some u s.t. $s = \mathbf{f}(u)$. By the definition of E', this means that there is a Z such that $\mathbf{image}(Z) \subseteq Y$ and $Z \in E(\emptyset)(s)$. Since $Y \subset \mathbf{image}(X)$ and $\mathbf{image}(Z) \subseteq Y$ it follows that $\mathbf{image}(Z) \subset \mathbf{image}(X)$. But it is a property of the **image** function that that implies that $Z \subset Y$, and this contradicts the assumption that $Z \in E(\emptyset)(s)$ and $X \in E^{nc}(\emptyset)(s)$.

In order to show that M' satisfies ϕ, we show that $M, \mathbf{f}(u) \models \gamma$ iff $M', u \models \gamma$ for $u \in S'$ and any γ, by induction in γ. All cases except $\gamma = [G]\phi$ are exactly as in the proof of Theorem 2 (see [14, Theorem 9] for a detailed inductive proof).

For the case that $\gamma = [G]\phi$, the inductive hypothesis is that for all ψ with $|\psi| < |[G]\phi|$, and any t, v with $t = \mathbf{f}(v)$, $M, t \models \psi$ iff $M', v \models \psi$. Given that for every v there is a unique t such that $t = \mathbf{f}(v)$, we can state this as $\{v : M', v \models \psi\} = \mathbf{image}(\psi^M)$, or $\psi^{M'} = \mathbf{image}(\psi^M)$.

First consider $G \neq N$. $M, s \models [G]\phi$ iff $\phi^M \in E(G)(s)$. Consider $\mathbf{image}(\phi^M)$. By the inductive hypothesis, $\phi^{M'} = \mathbf{image}(\phi^M)$. $\phi^M \in E(G)(s)$ holds iff $\phi^{M'} \in E'(G)(u)$ iff $M', u \models [G]\phi$.

$M, s \models [N]\phi$ iff $\phi^M \in E(N)(s)$ iff (*) $\neg\phi^M \notin E(\emptyset)(s)$ iff (as above) $\neg\phi^{M'} \notin E'(\emptyset)(u)$ iff $M', u \models [N]\phi$.

Explanation of (*): one direction E3, the other direction E5 and E1. ❑

Corollary 2 *For any CLD-formula ϕ, $\vdash_{CLD} \phi$ iff $\models \phi$.*

5 Completeness of Coalition Logic with both Common and Distributed Knowledge

In this section we consider the logic $\mathcal{CLCD}$, extending coalition logic with operators for individual knowledge, common knowledge and distributed knowledge.

The axiomatisation $CLCD$ is shown in Figure 4. It extends CL with standard axioms and rules for individual, common and distributed knowledge.

As usual, soundness can easily be shown.

Lemma 5 (Soundness) *For any $CLCD$-formula ϕ, $\vdash_{CLCD} \phi \Rightarrow \models \phi$.*

In the remainder of this section we show that $CLCD$ also is complete.

Theorem 4 *Any $CLCD$-consistent formula is satisfied in a finite pseudomodel.*

Proof The proof is identical to the proof of Lemma 4, with the addition of the inductive clause $\theta = C_G\psi$ as in the proof of Theorem 1. ❑

We can now use the same approach as in the case of $\mathcal{CLD}$.

Theorem 5 *If a $\mathcal{CLCD}$ formula is satisfied in a finite pseudomodel, it is satisfied in a model.*

Proof The proof goes exactly like the proof of Theorem 3, using Theorem 2. The definition of the model M' is identical to the definition in Theorem 3, as is the proof that it is a proper model. For the last part of the proof, i.e., showing that M' satisfies ϕ, note that the last clause in Theorem 2 holds for epistemic logic with both distributed and common knowledge. Thus, the proof is completed by only adding the inductive clause for $[G]\phi$, which is done in exactly the same way as in Theorem 3. ❑

Corollary 3 *For any $CLCD$-formula ϕ, $\vdash_{CLCD} \phi$ iff $\models \phi$.*

6 Conclusions

This papers solves several hitherto open problems, namely proving completeness of Coalition Logic extended with group knowledge modalities. The axioms for the epistemic modalities are the same as in the absence of the Coalition Logic axioms, however the completeness proofs require non-trivial combinations of techniques. The next step would be to look at complete axiomatisations of logics resulting from imposing some conditions on the interaction of coalitional ability and group knowledge (such as the examples in the Introduction), and obtaining results on the complexity of satisfiability problem for $\mathcal{CLC}$, $\mathcal{CLD}$ and $\mathcal{CLCD}$.

References

[1] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49:672–713, 2002.

[2] N. Belnap and M. Perloff. Seeing to it that: a canonical form for agentives. *Theoria*, 54:175–199, 1988.

[3] Jan Broersen. A complete stit logic for knowledge and action, and some of its applications. In Matteo Baldoni, Tran Cao Son, M. Birna van Riemsdijk, and Michael Winikoff, editors, *Declarative Agent Languages and Technologies VI, 6th International Workshop, DALT 2008, Estoril, Portugal, May 12, 2008, Revised Selected and Invited Papers*, volume 5397 of *Lecture Notes in Computer Science*, pages 47–59, 2009.

[4] Jan Broersen, Andreas Herzig, and Nicolas Troquard. A normal simulation of coalition logic and an epistemic extension. In Dov Samet, editor, *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK-2007), Brussels, Belgium, June 25-27, 2007*, pages 92–101, 2007.

[5] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* The MIT Press, Cambridge, Massachusetts, 1995.

[6] V. Goranko and G. van Drimmelen. Complete axiomatization and decidability of alternating-time temporal logic. *Theoretical Computer Science*, 353(1):93–117, 2006.

[7] Valentin Goranko, Wojciech Jamroga, and Govert van Drimmelen. Axiomatic systems for alternating time temporal epistemic logics (extended abstract).

[8] Valentin Goranko, Wojtek Jamroga, and Paolo Turrini. Strategic games and truly playable effectivity functions. In Tumer, Yolum, Sonenberg, and Stone, editors, *Proceedings of the 10th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2011)*, pages 727–734, Taipei, Taiwan, 2011.

[9] A. Herzig J. Broersen and N. Troquard. What groups do, can do, and know they can do: an analysis in normal modal logics. *Journal of Applied Non-Classical Logic*, 19(3):261–289, 2009.

[10] J.-J. Ch. Meyer and W. van der Hoek. *Epistemic Logic for AI and Computer Science.* Cambridge University Press: Cambridge, England, 1995.

[11] M. Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 12(1):149–166, 2002.

[12] Wiebe van der Hoek and Michael Wooldridge. Cooperation, knowledge and time: Alternating-time temporal epistemic logic and its applications. *Studia Logica*, 75:125–157, 2003.

[13] H.P. van Ditmarsch, W. van der Hoek, and B.P. Kooi. *Dynamic Epistemic Logic*, volume 337 of *Synthese Library.* Springer, 2007.

[14] Yi Wang and Thomas Ågotnes. Public announcement logic with distributed knowledge. Manuscript.

Prop Classical propositional logic

G1 $\neg[G]\bot$

G2 $[G]\top$

G3 $\neg[\emptyset]\neg\phi \rightarrow [N]\phi$

G4 $[G](\phi \wedge \psi) \rightarrow [G]\psi$

G5 $[G_1]\phi \wedge [G_2]\psi \rightarrow [G_1 \cup G_2](\phi \wedge \psi)$, if $G_1 \cap G_2 = \emptyset$

MP $\vdash_{CLD} \phi, \phi \rightarrow \psi \Rightarrow \vdash_{CLD} \psi$

RG $\vdash_{CLD} \phi \leftrightarrow \psi \Rightarrow \vdash_{CLD} [G]\phi \leftrightarrow [G]\psi$

K $K_i(\phi \rightarrow \psi) \rightarrow (K_i\phi \rightarrow K_i\psi)$

T $K_i\phi \rightarrow \phi$

4 $K_i\phi \rightarrow K_iK_i\phi$

5 $\neg K_i\phi \rightarrow K_i\neg K_i\phi$

RN $\vdash_{CLD} \phi \Rightarrow \vdash_{CLD} K_i\phi$

DK $D_G(\phi \rightarrow \psi) \rightarrow (D_G\phi \rightarrow D_G\psi)$

DT $D_G\phi \rightarrow \phi$

D4 $D_G\phi \rightarrow D_GD_G\phi$

D5 $\neg D_G\phi \rightarrow D_G\neg D_G\phi$

D1 $K_i\phi \leftrightarrow D_i\phi$

D2 $D_G\phi \rightarrow D_H\phi$, if $G \subseteq H$

Figure 3: CLD: axiomatisation of $\mathcal{CLD}$.

Prop Classical propositional logic

G1 $\neg[G]\bot$

G2 $[G]\top$

G3 $\neg[\emptyset]\neg\phi \rightarrow [N]\phi$

G4 $[G](\phi \wedge \psi) \rightarrow [G]\psi$

G5 $[G_1]\phi \wedge [G_2]\psi \rightarrow [G_1 \cup G_2](\phi \wedge \psi)$, if $G_1 \cap G_2 = \emptyset$

MP $\vdash_{CLCD} \phi, \phi \rightarrow \psi \Rightarrow \vdash_{CLCD} \psi$

RG $\vdash_{CLCD} \phi \leftrightarrow \psi \Rightarrow \vdash_{CLCD} [G]\phi \leftrightarrow [G]\psi$

K $K_i(\phi \rightarrow \psi) \rightarrow (K_i\phi \rightarrow K_i\psi)$

T $K_i\phi \rightarrow \phi$

4 $K_i\phi \rightarrow K_iK_i\phi$

5 $\neg K_i\phi \rightarrow K_i\neg K_i\phi$

RN $\vdash_{CLCD} \phi \Rightarrow \vdash_{CLCD} K_i\phi$

C1 $E_G\phi \leftrightarrow \bigwedge_{i\in G} K_i\phi$

C2 $C_G\phi \rightarrow E_G(\phi \wedge C_G\phi)$

RN $\vdash_{CLCD} \phi \Rightarrow \vdash_{CLCD} K_i\phi$

RC $\vdash_{CLCD} \phi \rightarrow E_G(\phi \wedge \psi) \Rightarrow \vdash_{CLCD} \phi \rightarrow C_G\psi$

DK $D_G(\phi \rightarrow \psi) \rightarrow (D_G\phi \rightarrow D_G\psi)$

DT $D_G\phi \rightarrow \phi$

D4 $D_G\phi \rightarrow D_GD_G\phi$

D5 $\neg D_G\phi \rightarrow D_G\neg D_G\phi$

D1 $K_i\phi \leftrightarrow D_i\phi$

D2 $D_G\phi \rightarrow D_H\phi$, if $G \subseteq H$

Figure 4: $CLCD$: axiomatisation of $\mathcal{CLCD}$.